Exercise 1

Suppose that $u = u(x,t)$ and $v = v(x,t)$ have partial derivatives related in the following way:

$$\frac{\partial u}{\partial t} = -\frac{\partial v}{\partial x} \quad \text{and} \quad \frac{\partial v}{\partial t} = -\frac{\partial u}{\partial x}.$$

Show that u and v are solutions of the wave equation (1) with $c = 1$.

Solution

Differentiate both sides of the equation on the left and right with respect to x and t, respectively.

$$\begin{cases} \dfrac{\partial u}{\partial t} = -\dfrac{\partial v}{\partial x} \\ \dfrac{\partial u}{\partial x} = -\dfrac{\partial v}{\partial t} \end{cases} \quad \rightarrow \quad \begin{cases} \dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial t}\right) = \dfrac{\partial}{\partial x}\left(-\dfrac{\partial v}{\partial x}\right) \\ \dfrac{\partial}{\partial t}\left(\dfrac{\partial u}{\partial x}\right) = \dfrac{\partial}{\partial t}\left(-\dfrac{\partial v}{\partial t}\right) \end{cases} \quad \rightarrow \quad \begin{cases} \dfrac{\partial^2 u}{\partial x \partial t} = -\dfrac{\partial^2 v}{\partial x^2} \\ \dfrac{\partial^2 u}{\partial t \partial x} = -\dfrac{\partial^2 v}{\partial t^2} \end{cases}$$

The mixed partial derivatives are equal by Clairaut's theorem.

$$-\frac{\partial^2 v}{\partial t^2} = -\frac{\partial^2 v}{\partial x^2}$$

Multiply both sides by -1.

$$\frac{\partial^2 v}{\partial t^2} = \frac{\partial^2 v}{\partial x^2}$$

Therefore, v satisfies the wave equation with $c = 1$. Now instead, differentiate both sides of the equation on the left and right with respect to t and x, respectively.

$$\begin{cases} \dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial t} \\ \dfrac{\partial v}{\partial t} = -\dfrac{\partial u}{\partial x} \end{cases} \quad \rightarrow \quad \begin{cases} \dfrac{\partial}{\partial t}\left(\dfrac{\partial v}{\partial x}\right) = \dfrac{\partial}{\partial t}\left(-\dfrac{\partial u}{\partial t}\right) \\ \dfrac{\partial}{\partial x}\left(\dfrac{\partial v}{\partial t}\right) = \dfrac{\partial}{\partial x}\left(-\dfrac{\partial u}{\partial x}\right) \end{cases} \quad \rightarrow \quad \begin{cases} \dfrac{\partial^2 v}{\partial t \partial x} = -\dfrac{\partial^2 u}{\partial t^2} \\ \dfrac{\partial^2 v}{\partial x \partial t} = -\dfrac{\partial^2 u}{\partial x^2} \end{cases}$$

The mixed partial derivatives are equal by Clairaut's theorem.

$$-\frac{\partial^2 u}{\partial t^2} = -\frac{\partial^2 u}{\partial x^2}$$

Multiply both sides by -1.

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

Therefore, u also satisfies the wave equation with $c = 1$.